(1+1)-Dimensional Methods for General Relativity

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Abstract

We present the (1+1)-dimensional method for studying general relativity of 4-dimensions. We first discuss the general formalism, and subsequently draw attention to the algebraically special class of space-times, following the Petrov classification. It is shown that this class of space-times can be described by the (1+1)-dimensional Yang-Mills action interacting with matter fields, with the spacial diffeomorphisms of the 2-surface as the gauge symmetry. The constraint appears polynomial in part, whereas the non-polynomial part is a non-linear sigma model type in (1+1)-dimensions. It is also shown that the representations of w_∞-gravity appear naturally as special cases of this description, and we discuss briefly the w_∞-geometry in term of the fibre bundle.

1. Introduction

For past years many 2-dimensional field theories have been intensively studied as laboratories for many theoretical issues, due to great mathematical simplicities that often exist in 2-dimensional systems. Recently these 2-dimensional field theories have received considerable attention, for different reasons, in connection with general relativistic systems of 4-dimensions, such as self-dual spaces [1] and the black-hole space-times [2, 3]. These 2-dimensional formulations of self-dual spaces and black-hole space-times of allow, in principle, many 2-dimensional field theoretic methods developed in the past relevant for the description of the physics of 4-dimensions. This raises an intriguing question as to whether it is also possible to describe general relativity itself as a 2-dimensional field theory. Recently we have shown that such a description is indeed possible, and obtained, at least formally, the corresponding (1+1)-dimensional action principle based on the (2+2)-decomposition of general space-times[1] [4]. In particular, the algebraically special class of space-times (the

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[1] Here we are viewing space-times of 4-dimensions as locally fibrated, $M_{1+1} \times N_2$, with M_{1+1} as the base manifold of signature $(-,+)$ and N_2 as the 2-dimensional fibre space of signature $(+,+)$.

Petrov type II), following the Petrov classification [5], was studied as an illustration from this perspective, and the (1+1)-dimensional action principle and the constraints for this class were identified [6]. In this (2+2)-decomposition general relativity shows up as a (1+1)-dimensional gauge theory interacting with (1+1)-dimensional matter fields, with the *minimal* coupling to the gauge fields, where the gauge symmetry is the diffeomorphisms of the fibre of 2-spacial dimensions. In this article we shall review our recent attempts of the (1+1)-dimensional formulation of general relativity. This article is organized as follows. In section 2, we present the general formalism of (2+2)-decomposition of general relativity, and establish the corresponding (1+1)-dimensional action principle.

In section 3, we draw attention to the algebraically special class of space-times [5, 7], following the Petrov classification, and present the (1+1)-dimensional action principle for this entire class of space-times. We shall show that the spacial diffeomorphisms of the 2-surface becomes the gauge fixing condition in this description. The constraint is polynomial in part, whereas the non-polynomial term is a non-linear sigma model type in (1+1)-dimensions. As such, this formulation might render the problem of the constraints of general relativity manageable, at least formally.

In section 4, we discuss the realizations of the so-called w_∞-gravity as special cases of this description. We find the fibre bundle as the natural framework for the geometric description of w_∞-gravity, whose geometric understanding was lacking so far [8, 9]. In this picture the local gauge fields for w_∞-gravity are identified as the connections valued in the infinite dimensional Lie algebra associated with the area-preserving diffeomorphisms of the 2-dimensional fibre. Due to this picture of w_∞-geometry, we are able to construct field theoretic realizations of w_∞-gravity in a straightforward way. In section 5, we summarize this review and discuss a few problems for the future investigations.

2. $(2+2)$-decomposition of general relativity

Consider a 4-dimensional manifold $P_4 \simeq M_{1+1} \times N_2$, equipped with a metric g_{AB} $(A, B, \cdots = 0,1,2,3)$[2]. Let $\partial_\mu = \partial/\partial x^\mu$ $(\mu, \nu, \cdots = 0,1)$ and $\partial_a = \partial/\partial y^a$ $(a, b, \cdots = 2,3)$ be a coordinate basis of M_{1+1} and N_2, respectively, and choose $\partial_A = (\partial_\mu, \partial_a)$ as a coordinate basis of P_4. In this basis the most general metric on P_4 can be written as [10]

$$ds^2 = \phi_{ab} dy^a dy^b + \Big(\gamma_{\mu\nu} + \phi_{ab} A_\mu^{\ a} A_\nu^{\ b}\Big) dx^\mu dx^\nu + 2\phi_{ab} A_\mu^{\ b} dx^\mu dy^a. \tag{2.1}$$

[2]From here on, we shall distinguish the two manifolds by their signatures to avoid confusion. Namely, M_{1+1} shall be referred to as the (1+1)-dimensional manifold and N_2 as 2-dimensional manifold.

Formally this is quite similar to the 'dimensional reduction' in Kaluza-Klein theory, where N_2 is regarded as the 'internal' fibre [3] and M_{1+1} as the 'space-time'. In the standard Kaluza-Klein reduction one assumes a restriction on the metric, namely, an isometry condition, to make ${A_\mu}^a$ a gauge field associated with the isometry group. Here, however, we do not assume any isometry condition, and allow all the fields to depend arbitrarily on both x^μ and y^a. Nevertheless ${A_\mu}^a(x,y)$ can still be identified as a connection, but now associated with an infinite dimensional diffeomorphism group diffN_2. To show this, let us consider the following diffeomorphism of N_2,

$$y^{'a} = y^{'a}(y^b, x^\mu), \qquad x^{'\mu} = x^\mu. \tag{2.2}$$

Under these transformations, we find

$$\gamma'_{\mu\nu}(y', x) = \gamma_{\mu\nu}(y, x), \tag{2.3a}$$

$$\phi'_{ab}(y', x) = \frac{\partial y^c}{\partial y^{'a}} \frac{\partial y^d}{\partial y^{'b}} \phi_{cd}(y, x), \tag{2.3b}$$

$${A_\mu}^{'a}(y', x) = \frac{\partial y^{'a}}{\partial y^c} {A_\mu}^c(y, x) - \partial_\mu y^{'a}. \tag{2.3c}$$

For the corresponding infinitesimal variations such that

$$\delta y^a = \xi^a(y^b, x^\mu), \qquad \delta x^\mu = 0, \tag{2.4}$$

(2.3) become

$$\begin{aligned} \delta\gamma_{\mu\nu} &= -[\xi, \gamma_{\mu\nu}] = -\pounds_\xi \gamma_{\mu\nu} = -\xi^c \partial_c \gamma_{\mu\nu}, & (2.5a) \\ \delta\phi_{ab} &= -[\xi, \phi]_{ab} = -\pounds_\xi \phi_{ab} & \\ &= -\xi^c \partial_c \phi_{ab} - (\partial_a \xi^c)\phi_{cb} - (\partial_b \xi^c)\phi_{ac}, & (2.5b) \\ \delta {A_\mu}^a &= -\partial_\mu \xi^a + [A_\mu,\ \xi]^a = -\partial_\mu \xi^a + \pounds_{A_\mu} \xi^a & \\ &= -\partial_\mu \xi^a + (A_\mu^c \partial_c \xi^a - \xi^c \partial_c {A_\mu}^a), & (2.5c) \end{aligned}$$

where $\pounds_\xi$ represents the Lie derivative along the vector fields $\xi = \xi^a \partial_a$, and acts only on the 'internal' indices a, b, etc. Notice that the Lie derivative, an *infinite* dimensional generalization of the finite dimensional matrix commutators, appears naturally. Clearly (2.4) defines a gauge transformation which leaves the line element (2.1) invariant. Associated with this gauge transformation, the *covariant* derivative D_μ is defined by

$$D_\mu = \partial_\mu - \pounds_{A_\mu}, \tag{2.6}$$

where the Lie derivative is taken along the vector field $A_\mu = {A_\mu}^a \partial_a$. With this definition, we have

$$\delta {A_\mu}^a = -D_\mu \xi^a, \tag{2.7}$$

[3] For the algebraically special class of space-times we shall consider in section 3, the fibre space N_2 may be interpreted as the physical transverse wave-surface [5].

which clearly indicates that $A_\mu^{\ a}$ is the gauge field valued in the infinite dimensional Lie algebra associated with the diffeomorphisms of N_2. Moreover the transformation properties (2.3a) and (2.3b) show that $\gamma_{\mu\nu}$ and ϕ_{ab} are a scalar and tensor field, respectively, under diffN_2. The field strength $F_{\mu\nu}^{\ \ a}$ corresponding to $A_\mu^{\ a}$ can now be defined as

$$[D_\mu, D_\nu] = -F_{\mu\nu}^{\ \ a}\partial_a = -\{\partial_\mu A_\nu^{\ a} - \partial_\nu A_\mu^{\ a} - [A_\mu, A_\nu]^a\}\partial_a. \tag{2.8}$$

Notice that the field strength transforms covariantly under the infinitesimal transformation (2.4),

$$\delta F_{\mu\nu}^{\ \ a} = -[\xi, F_{\mu\nu}]^a = -\pounds_\xi F_{\mu\nu}^{\ \ a}. \tag{2.9}$$

To find the (1+1)-dimensional action principle of general relativity, we must compute the scalar curvature of space-times in the (2+2)-decomposition. For this purpose it is convenient to introduce the following non-coordinate basis $\hat{\partial}_A = (\hat{\partial}_\mu, \hat{\partial}_a)$ where [11]

$$\hat{\partial}_\mu \equiv \partial_\mu - A_\mu^{\ a}(x,y)\partial_a, \qquad \hat{\partial}_a \equiv \partial_a\ . \tag{2.10}$$

From the definition we have

$$[\hat{\partial}_A, \hat{\partial}_B] = f_{AB}^{\ \ \ C}(x,y)\hat{\partial}_C, \tag{2.11}$$

where the structure coefficients $f_{AB}^{\ \ \ C}$ are given by

$$\begin{aligned} &f_{\mu\nu}^{\ \ a} = -F_{\mu\nu}^{\ \ a}, \\ &f_{\mu a}^{\ \ b} = -f_{a\mu}^{\ \ b} = \partial_a A_\mu^{\ b}, \\ &f_{AB}^{\ \ \ C} = 0, \qquad \text{otherwise.} \end{aligned} \tag{2.12}$$

The virtue of this basis is that it brings the metric (2.1) into a block diagonal form

$$g_{AB} = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \phi_{ab} \end{pmatrix}, \tag{2.13}$$

which drastically simplifies the computation of the scalar curvature. In this basis the Levi-Civita connections are given by

$$\Gamma_{AB}^{\ \ \ C} = \frac{1}{2}g^{CD}(\hat{\partial}_A g_{BD} + \hat{\partial}_B g_{AD} - \hat{\partial}_D g_{AB}) + \frac{1}{2}g^{CD}(f_{ABD} - f_{BDA} - f_{ADB}), \tag{2.14}$$

where $f_{ABC} = g_{CD} f_{AB}^{\ \ \ D}$. For completeness, we present the connection coefficients in components,

$$\begin{aligned} \Gamma_{\mu\nu}^{\ \ \alpha} &= \frac{1}{2}\gamma^{\alpha\beta}\Big(\hat{\partial}_\mu\gamma_{\nu\beta} + \hat{\partial}_\nu\gamma_{\mu\beta} - \hat{\partial}_\beta\gamma_{\mu\nu}\Big), \\ \Gamma_{\mu\nu}^{\ \ a} &= -\frac{1}{2}\phi^{ab}\partial_b\gamma_{\mu\nu} - \frac{1}{2}F_{\mu\nu}^{\ \ a}, \end{aligned}$$

$$
\begin{aligned}
&\Gamma_{\mu a}{}^{\nu} = \Gamma_{a\mu}{}^{\nu} = \frac{1}{2}\gamma^{\nu\alpha}\partial_a\gamma_{\mu\alpha} + \frac{1}{2}\gamma^{\nu\alpha}\phi_{ab}F_{\mu\alpha}{}^{b},\\
&\Gamma_{\mu a}{}^{b} = \frac{1}{2}\phi^{bc}\hat{\partial}_\mu\phi_{ac} + \frac{1}{2}\partial_a A_\mu{}^{b} - \frac{1}{2}\phi^{bc}\phi_{ae}\partial_c A_\mu{}^{e},\\
&\Gamma_{a\mu}{}^{b} = \frac{1}{2}\phi^{bc}\hat{\partial}_\mu\phi_{ac} - \frac{1}{2}\partial_a A_\mu{}^{b} - \frac{1}{2}\phi^{bc}\phi_{ae}\partial_c A_\mu{}^{e},\\
&\Gamma_{ab}{}^{\mu} = -\frac{1}{2}\gamma^{\mu\nu}\hat{\partial}_\nu\phi_{ab} + \frac{1}{2}\gamma^{\mu\nu}\phi_{ac}\partial_b A_\nu{}^{c} + \frac{1}{2}\gamma^{\mu\nu}\phi_{bc}\partial_a A_\nu{}^{c},\\
&\Gamma_{ab}{}^{c} = \frac{1}{2}\phi^{cd}\Big(\partial_a\phi_{bd} + \partial_b\phi_{ad} - \partial_d\phi_{ab}\Big).
\end{aligned}
\tag{2.15}
$$

For later purposes it is useful to have the following identities,

$$
\Gamma_{\alpha\mu}{}^{\alpha} = \frac{1}{2}\gamma^{\alpha\beta}\hat{\partial}_\mu\gamma_{\alpha\beta}, \qquad \Gamma_{a\mu}{}^{a} = \frac{1}{2}\phi^{ab}\hat{\partial}_\mu\phi_{ab} - \partial_a A_\mu{}^{a}, \tag{2.16a}
$$

$$
\Gamma_{\beta a}{}^{\beta} = \frac{1}{2}\gamma^{\alpha\beta}\partial_a\gamma_{\alpha\beta}, \qquad \Gamma_{ba}{}^{b} = \frac{1}{2}\phi^{bc}\partial_a\phi_{bc}. \tag{2.16b}
$$

The curvature tensors are defined as

$$
\begin{aligned}
&R_{ABC}{}^{D} = \hat{\partial}_A\Gamma_{BC}{}^{D} - \hat{\partial}_B\Gamma_{AC}{}^{D} + \Gamma_{AE}{}^{D}\Gamma_{BC}{}^{E} - \Gamma_{BE}{}^{D}\Gamma_{AC}{}^{E} - f_{AB}{}^{E}\Gamma_{EC}{}^{D},\\
&R_{AC} = R_{ABC}{}^{B}, \qquad R = g^{AC}R_{AC}.
\end{aligned}
\tag{2.17}
$$

Explicitly, the scalar curvature R is given by

$$
R = \gamma^{\mu\nu}(R_{\mu\alpha\nu}{}^{\alpha} + R_{\mu a\nu}{}^{a}) + \phi^{ab}(R_{acb}{}^{c} + R_{a\mu b}{}^{\mu}) \ , \tag{2.18}
$$

which becomes, after a lengthy computation,

$$
\begin{aligned}
R \ = \ & \gamma^{\mu\nu}R_{\mu\nu} + \phi^{ac}R_{ac} + \frac{1}{4}\phi_{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}F_{\mu\alpha}{}^{a}F_{\nu\beta}{}^{b}\\
&+\frac{1}{4}\gamma^{\mu\nu}\phi^{ab}\phi^{cd}\Big\{(D_\mu\phi_{ac})(D_\nu\phi_{bd}) - (D_\mu\phi_{ab})(D_\nu\phi_{cd})\Big\}\\
&+\frac{1}{4}\phi^{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}\Big\{(\partial_a\gamma_{\mu\alpha})(\partial_b\gamma_{\nu\beta}) - (\partial_a\gamma_{\mu\nu})(\partial_b\gamma_{\alpha\beta})\Big\} + \nabla_A j^{A},
\end{aligned}
\tag{2.19}
$$

where $R_{\mu\nu}$ and R_{ac} are defined by

$$
R_{\mu\nu} = \hat{\partial}_\mu\Gamma_{\alpha\nu}{}^{\alpha} - \hat{\partial}_\alpha\Gamma_{\mu\nu}{}^{\alpha} + \Gamma_{\mu\beta}{}^{\alpha}\Gamma_{\alpha\nu}{}^{\beta} - \Gamma_{\beta\alpha}{}^{\beta}\Gamma_{\mu\nu}{}^{\alpha}, \tag{2.20a}
$$

$$
R_{ac} = \partial_a\Gamma_{bc}{}^{b} - \partial_b\Gamma_{ac}{}^{b} + \Gamma_{ad}{}^{b}\Gamma_{bc}{}^{d} - \Gamma_{db}{}^{d}\Gamma_{ac}{}^{b}. \tag{2.20b}
$$

The last term in (2.19) is given by

$$
\nabla_A j^{A} = \nabla_\mu j^{\mu} + \nabla_a j^{a}, \tag{2.21a}
$$

$$
\nabla_\mu j^{\mu} = \Big(\hat{\partial}_\mu + \Gamma_{\alpha\mu}{}^{\alpha} + \Gamma_{c\mu}{}^{c}\Big)j^{\mu}, \tag{2.21b}
$$

$$
\nabla_a j^{a} = \Big(\partial_a + \Gamma_{ca}{}^{c} + \Gamma_{\alpha a}{}^{\alpha}\Big)j^{a}, \tag{2.21c}
$$

where j^μ and j^a are given by

$$j^\mu = \gamma^{\mu\nu}\Big(\phi^{ab}\hat{\partial}_\nu\phi_{ab} - 2\partial_a A_\nu{}^a\Big), \qquad j^a = \phi^{ab}\gamma^{\mu\nu}\partial_b\gamma_{\mu\nu}. \tag{2.22}$$

That $\nabla_A j^A$ is a surface term in the action integral can be seen easily, using (2.16). For instance let us show that $\sqrt{-\gamma}\sqrt{\phi}\nabla_\mu j^\mu$ is a surface term, where $\gamma = \det\gamma_{\mu\nu}$ and $\phi = \det\phi_{ab}$. From (2.21b) we have

$$\sqrt{-\gamma}\sqrt{\phi}\nabla_\mu j^\mu = \sqrt{-\gamma}\sqrt{\phi}\Big[\partial_\mu j^\mu - A_\mu{}^a\partial_a j^\mu + \Big(\Gamma_{\alpha\mu}{}^\alpha + \Gamma_{c\mu}{}^c\Big)j^\mu\Big]. \tag{2.23}$$

The first term in the r.h.s. of (2.23) can be written as

$$\sqrt{-\gamma}\sqrt{\phi}\partial_\mu j^\mu = -\frac{1}{2}\sqrt{-\gamma}\sqrt{\phi}\Big(\gamma^{\alpha\beta}\partial_\mu\gamma_{\alpha\beta} + \phi^{ab}\partial_\mu\phi_{ab}\Big)j^\mu + \partial_\mu\Big(\sqrt{-\gamma}\sqrt{\phi}j^\mu\Big), \tag{2.24}$$

and for the second term, we have

$$\sqrt{-\gamma}\sqrt{\phi}A_\mu{}^a\partial_a j^\mu = -\sqrt{-\gamma}\sqrt{\phi}\Big[\Big\{A_\mu{}^a(\Gamma_{\alpha a}{}^\alpha + \Gamma_{ba}{}^b) + \partial_a A_\mu{}^a\Big\}j^\mu\Big] + \partial_a\Big(\sqrt{-\gamma}\sqrt{\phi}A_\mu{}^a j^\mu\Big). \tag{2.25}$$

The last two terms in the r. h. s. of (2.23) becomes, using (2.16),

$$\begin{aligned}\sqrt{-\gamma}\sqrt{\phi}\Big(\Gamma_{\alpha\mu}{}^\alpha + \Gamma_{c\mu}{}^c\Big)j^\mu &= \sqrt{-\gamma}\sqrt{\phi}\Big(\frac{1}{2}\gamma^{\alpha\beta}\partial_\mu\gamma_{\alpha\beta} - A_\mu{}^a\Gamma_{\alpha a}{}^\alpha + \frac{1}{2}\phi^{ab}\partial_\mu\phi_{ab} \\ &\quad - A_\mu{}^a\Gamma_{ba}{}^b - \partial_a A_\mu{}^a\Big)j^\mu.\end{aligned} \tag{2.26}$$

Putting (2.24), (2.25), and (2.26) into (2.23), we find that it is a total divergence term,

$$\sqrt{-\gamma}\sqrt{\phi}\nabla_\mu j^\mu = \partial_\mu\Big(\sqrt{-\gamma}\sqrt{\phi}j^\mu\Big) - \partial_a\Big(\sqrt{-\gamma}\sqrt{\phi}A_\mu{}^a j^a\Big), \tag{2.27}$$

which we may ignore. Similarly, $\sqrt{-\gamma}\sqrt{\phi}\nabla_a j^a$ is also a surface term. This altogether shows that $\sqrt{-\gamma}\sqrt{\phi}\nabla_A j^A$ is indeed a total divergence term.

At this point it is important to notice the followings. First, $D_\mu\phi_{ab}$, written as

$$\begin{aligned} D_\mu\phi_{ab} &= \partial_\mu\phi_{ab} - \pounds_{A_\mu}\phi_{ab} \\ &= \partial_\mu\phi_{ab} - \Big\{A_\mu{}^c(\partial_c\phi_{ab}) + (\partial_a A_\mu{}^c)\phi_{cb} + (\partial_b A_\mu{}^c)\phi_{ac}\Big\}\end{aligned} \tag{2.28}$$

indeed transforms covariantly under the infinitesimal diffeomorphism (2.4),

$$\delta(D_\mu\phi_{ab}) = -\pounds_\xi(D_\mu\phi_{ab}) = -[\xi, D_\mu\phi]_{ab}. \tag{2.29}$$

Second, the derivative $\hat{\partial}_\mu$, when applied to $\gamma_{\mu\nu}$, becomes the covariant derivative

$$\hat{\partial}_\mu\gamma_{\alpha\beta} = \partial_\mu\gamma_{\alpha\beta} - \pounds_{A_\mu}\gamma_{\alpha\beta} = D_\mu\gamma_{\alpha\beta}, \tag{2.30}$$

so that $\hat{\partial}_\mu \gamma_{\alpha\beta}$ transforms covariantly

$$\delta(\hat{\partial}_\mu \gamma_{\alpha\beta}) = -\pounds_\xi (D_\mu \gamma_{\alpha\beta}) = -[\xi, D_\mu \gamma_{\alpha\beta}]. \tag{2.31}$$

These observations play an important role when we discuss the gauge invariance of the theory under diffN_2. It is worth mentioning here that, from (2.20a) and (2.30), $R_{\mu\nu}$ becomes the 'covariantized' Ricci tensor

$$R_{\mu\nu} = D_\mu \Gamma_{\alpha\nu}{}^\alpha - D_\alpha \Gamma_{\mu\nu}{}^\alpha + \Gamma_{\mu\beta}{}^\alpha \Gamma_{\alpha\nu}{}^\beta - \Gamma_{\beta\alpha}{}^\beta \Gamma_{\mu\nu}{}^\alpha, \tag{2.32}$$

as $\Gamma_{\mu\nu}{}^\alpha$'s do not involve the 'internal' indices a, b, etc. Thus we might call $\gamma^{\mu\nu} R_{\mu\nu}$ as the 'gauged' gravity action in (1+1)-dimensions [12].

With the scalar curvature at hand, one can easily write down the lagrangian for the Einstein-Hilbert action on P_4. From (2.19) we have

$$\begin{aligned} \mathcal{L}_2 \;=\; & -\sqrt{-\gamma}\sqrt{\phi}\Big[\gamma^{\mu\nu} R_{\mu\nu} + \phi^{ab} R_{ab} + \frac{1}{4}\phi_{ab} F_{\mu\nu}{}^a F^{\mu\nu b} \\ & + \frac{1}{4}\gamma^{\mu\nu}\phi^{ab}\phi^{cd}\Big\{(D_\mu \phi_{ac})(D_\nu \phi_{bd}) - (D_\mu \phi_{ab})(D_\nu \phi_{cd})\Big\} \\ & + \frac{1}{4}\phi^{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}\Big\{(\partial_a \gamma_{\mu\alpha})(\partial_b \gamma_{\nu\beta}) - (\partial_a \gamma_{\mu\nu})(\partial_b \gamma_{\alpha\beta})\Big\}\Big], \end{aligned} \tag{2.33}$$

neglecting the total divergence term (2.27). Clearly the action principle describes a (1+1)-dimensional field theory which is invariant under the gauge transformation of diffN_2, as the gauge field $A_\mu{}^a$ couples *minimally* to both $\gamma_{\mu\nu}$ and ϕ_{ab}. Therefore each term in (2.33) is invariant under diffN_2. To understand the physical contents of the theory we notice the followings. First, unlike the ordinary gravity, the metric $\gamma_{\mu\nu}$ of M_{1+1} here is 'charged', because it couples to $A_\mu{}^a$ (with the coupling constant 1). Second, the metric ϕ_{ab} of N_2 can be identified as a non-linear sigma field, whose self-interaction potential is determined by the scalar curvature $\phi^{ab} R_{ab}$ of N_2. The theory therefore describes a gauge theory of diffN_2 interacting with the 'gauged' gravity and the non-linear sigma field on M_{1+1}.

3. Algebraically special class of space-times

In contrast to the cases of the self-dual spaces and black-hole space-times, the (1+1)-dimensional action principle for general space-times, as we derived in the previous section, appears to be rather formal and consequently, of little practical use. In this section we therefore draw attention following the Petrov classification to a specific class of space-times, namely, the algebraically special class, and interpret the entire class from the (1+1)-dimensional point of view. It turns out that space-times of this class can be formulated as (1+1)-dimensional field theory in a remarkably simple form.

Let us consider a class of space-times that contain a twist-free null vector field k^A. These space-times belong to the algebraically special class of space-times, according to the Petrov classification. This class of space-times is rather broad, since most of the known exact solutions of the Einstein's equations are algebraically special. Being twist-free, the null vector field may be chosen to be a gradient field, so that $k_A = \partial_A u$ for some function u. The null hypersurface N_2 defined by $u =$ constant spans the 2-dimensional subspace for which we introduce two space-like coordinates y^a. The general line element for this class has the form [5, 7]

$$ds^2 = \phi_{ab} dy^a dy^b - 2du(dv + m_a dy^a + H du), \tag{3.1}$$

where v is the affine parameter, and ϕ_{ab}, m_a and H are functions of all of the four coordinates (u, v, y^a), as we assume no Killing vector fields.

For the class of space-times (3.1), we shall find the (1+1)-dimensional action principle defined on the (u, v)-surface. For this purpose let us first introduce the 'light-cone' coordinates (u, v) such that

$$u = \frac{1}{\sqrt{2}}(x^0 + x^1), \qquad v = \frac{1}{\sqrt{2}}(x^0 - x^1), \tag{3.2}$$

and define ${A_u}^a$ and ${A_v}^a$

$${A_u}^a = \frac{1}{\sqrt{2}}({A_0}^a + {A_1}^a), \qquad {A_v}^a = \frac{1}{\sqrt{2}}({A_0}^a - {A_1}^a). \tag{3.3}$$

For $\gamma_{\mu\nu}$, we assume the Polyakov ansatz [13]

$$\gamma_{\mu\nu} = \begin{pmatrix} -2h & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 2h \end{pmatrix}, \qquad (\det\gamma_{\mu\nu} = -1), \tag{3.4}$$

in the (u, v)-coordinates. Then the line element (2.1) becomes

$$\begin{aligned} ds^2 \; = \; & \phi_{ab} dy^a dy^b - 2du dv - 2h(du)^2 + \phi_{ab}({A_u}^a du + {A_v}^a dv)({A_u}^b du + {A_v}^b dv) \\ & + 2\phi_{ab}({A_u}^a du + {A_v}^a dv) dy^b. \end{aligned} \tag{3.5}$$

If we choose the 'light-cone' gauge [4] ${A_v}^a = 0$, then this becomes

$$ds^2 = \phi_{ab} dy^a dy^b - 2\, du\Big[dv - \phi_{ab} {A_u}^b dy^a + \Big(h - \frac{1}{2}\phi_{ab}{A_u}^a {A_u}^b\Big) du\Big]. \tag{3.6}$$

A comparison of (3.1) and (3.6) tells us that if the following identifications

$$m_a = -\phi_{ab}{A_u}^b, \qquad H = h - \frac{1}{2}\phi_{ab}{A_u}^a{A_u}^b \tag{3.7}$$

[4]Here we are referring to the disposable gauge degrees of freedom in the action. There could be topological obstruction against globalizing this choice, as the general coordinate transformation of N_2 corresponds to the gauge transformation.

are made, then the two line elements are the same. This shows that the Polyakov ansatz (3.4) amounts to the restriction (modulo the gauge choice $A_v{}^a = 0$) to the algebraically special class of space-times that contain a twist-free null vector field.

Let us now examine the transformation properties of h, ϕ_{ab}, $A_u{}^a$, and $A_v{}^a$ under the diffeomorphism of N_2,

$$y'^a = y'^a(y^b, u, v), \qquad u' = u, \qquad v' = v. \tag{3.8}$$

Under these transformations, we find that

$$h'(y', u, v) = h(y, u, v), \tag{3.9a}$$

$$\phi'_{ab}(y', u, v) = \frac{\partial y^c}{\partial y'^a}\frac{\partial y^d}{\partial y'^b}\phi_{cd}(y, u, v), \tag{3.9b}$$

$$A_u{}'^a(y', u, v) = \frac{\partial y'^a}{\partial y^c}A_u{}^c(y, u, v) - \partial_u y'^a, \tag{3.9c}$$

$$A_v{}'^a(y', u, v) = -\partial_v y'^a, \tag{3.9d}$$

which become, under the infinitesimal variations, $\delta y^a = \xi^a(y, u, v)$ and $\delta x^\mu = 0$,

$$\delta h = -[\xi, h] = -\xi^a \partial_a h, \tag{3.10a}$$

$$\delta \phi_{ab} = -[\xi, \phi]_{ab} = -\xi^c \partial_c \phi_{ab} - (\partial_a \xi^c)\phi_{cb} - (\partial_b \xi^c)\phi_{ac}, \tag{3.10b}$$

$$\delta A_u{}^a = -D_u \xi^a = -\partial_u \xi^a + [A_u, \xi]^a, \tag{3.10c}$$

$$\delta A_v{}^a = -\partial_v \xi^a. \tag{3.10d}$$

This shows that h and ϕ_{ab} are a scalar and tensor field, respectively, and $A_u{}^a$ and $A_v{}^a$ are the gauge fields valued in the infinite dimensional Lie algebra associated with the group of diffeomorphisms of N_2. That $A_v{}^a$ is a pure gauge is clear, as it depends on the gauge function ξ^a only. Therefore it can be always set to zero, at least locally, by a suitable coordinate transformation (3.8). To maintain the explicit gauge invariance, however, we shall work with the line element (3.5) in the following, with the understanding that $A_v{}^a$ is a pure gauge.

Let us now proceed to write down the action principle for (3.5) in terms of the fields h, ϕ_{ab}, $A_u{}^a$, and $A_v{}^a$. For this purpose, it is convenient to decompose the 2-dimensional metric ϕ_{ab} into the conformal classes

$$\phi_{ab} = \Omega \rho_{ab}, \qquad (\Omega > 0 \ \text{ and } \ \det \rho_{ab} = 1). \tag{3.11}$$

The kinetic term K of ϕ_{ab} in (2.33) then becomes

$$\begin{aligned} K &\equiv \frac{1}{4}\sqrt{-\gamma}\sqrt{\phi}\gamma^{\mu\nu}\phi^{ab}\phi^{cd}\Big\{(D_\mu\phi_{ac})(D_\nu\phi_{bd}) - (D_\mu\phi_{ab})(D_\nu\phi_{cd})\Big\} \\ &= -\frac{(D_\mu\Omega)^2}{2\Omega} + \frac{1}{4}\Omega\gamma^{\mu\nu}\rho^{ab}\rho^{cd}(D_\mu\rho_{ac})(D_\nu\rho_{bd}) \\ &= -\frac{1}{2}\mathrm{e}^\sigma(D_\mu\sigma)^2 + \frac{1}{4}\mathrm{e}^\sigma\gamma^{\mu\nu}\rho^{ab}\rho^{cd}(D_\mu\rho_{ac})(D_\nu\rho_{bd}), \end{aligned} \tag{3.12}$$

where we defined σ by $\sigma = \ln\Omega$, and the covariant derivatives $D_\mu\Omega$, $D_\mu\rho_{ab}$, and $D_\mu\sigma$ are

$$D_\mu\Omega = \partial_\mu\Omega - A_\mu{}^a\partial_a\Omega - (\partial_a A_\mu{}^a)\Omega, \tag{3.13a}$$
$$D_\mu\rho_{ab} = \partial_\mu\rho_{ab} - [A_\mu, \rho]_{ab} + (\partial_c A_\mu{}^c)\rho_{ab}, \tag{3.13b}$$
$$D_\mu\sigma = \partial_\mu\sigma - A_\mu{}^a\partial_a\sigma - \partial_a A_\mu{}^a, \tag{3.13c}$$

respectively, where $[A_\mu, \rho]_{ab}$ is given by

$$[A_\mu, \rho]_{ab} = A_\mu{}^c\partial_c\rho_{ab} + (\partial_a A_\mu{}^c)\rho_{cb} + (\partial_b A_\mu{}^c)\rho_{ac}. \tag{3.14}$$

The inclusion of the divergence term $\partial_a A_\mu{}^a$ in (3.13) is necessary to ensure (3.13) transform covariantly (as the tensor fields) under diffN_2, since Ω and ρ_{ab} are the tensor densities of weight -1 and $+1$, respectively. Using the ansatz (3.4), the kinetic term (3.12) becomes

$$\begin{aligned} K &= \mathrm{e}^\sigma(D_+\sigma)(D_-\sigma) - \frac{1}{2}\mathrm{e}^\sigma\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) \\ &\quad -h\mathrm{e}^\sigma\Big\{(D_-\sigma)^2 - \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\Big\}, \end{aligned} \tag{3.15}$$

where $+(-)$ stands for $u(v)$. The Polyakov ansatz (3.4) simplifies enormously the remaining terms in the action (2.33), as we now show. Let us first notice that $\det\gamma_{\mu\nu} = -1$. Therefore the term

$$\sqrt{-\gamma}\sqrt{\phi}\phi^{ac}R_{ac} = \sqrt{\phi}\phi^{ac}R_{ac} \tag{3.16}$$

can be removed from the action being a surface term. Moreover, since we have

$$\gamma^{\mu\nu}\partial_a\gamma_{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\partial_a\sqrt{-\gamma} = 0, \tag{3.17}$$

the last term in the action (2.33) vanishes. Furthermore, one can easily verify that

$$\begin{aligned} \phi^{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}(\partial_a\gamma_{\mu\alpha})(\partial_b\gamma_{\nu\beta}) &= \phi^{ab}(\partial_a\gamma_{++})\gamma^{+-}(\partial_b\gamma_{-\alpha})\gamma^{\alpha+} \\ &= 0, \end{aligned} \tag{3.18}$$

since $\partial_b\gamma_{-\alpha} = 0$. The only remaining terms that contribute to the action (2.33) are thus the (1+1)-dimensional Yang-Mills action and the 'gauged' gravity action. The Yang-Mills action becomes

$$\frac{1}{4}\phi_{ab}F_{\mu\nu}{}^a F^{\mu\nu b} = -\frac{1}{2}\mathrm{e}^\sigma\rho_{ab}F_{+-}{}^a F_{+-}{}^b. \tag{3.19}$$

To express the 'gauged' Ricci scalar $\gamma^{\mu\nu}R_{\mu\nu}$ in terms of h and $A_v{}^a$, etc., we have to compute the Levi-Civita connections first. They are given by

$$\begin{aligned} &\Gamma_{++}{}^{+} = -D_-h, \qquad \Gamma_{++}{}^{-} = D_+h + 2hD_-h, \\ &\Gamma_{+-}{}^{-} = \Gamma_{-+}{}^{-} = D_-h, \end{aligned} \tag{3.20}$$

and vanishing otherwise. Thus the 'gauged' Ricci tensor becomes

$$R_{+-} = R_{-+} = -D_-^2 h, \qquad R_{--} = 0. \tag{3.21}$$

From (3.4) and (3.21), the 'gauged' Ricci scalar $\gamma^{\mu\nu} R_{\mu\nu}$ is given by

$$\gamma^{\mu\nu} R_{\mu\nu} = 2\gamma^{+-} R_{+-} = 2D_-^2 h, \tag{3.22}$$

since $\gamma^{++} = R_{--} = 0$. Putting together (3.15), (3.19), and (3.22) into (2.33), the action becomes

$$\begin{aligned} \mathcal{L}_2 &= -\frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{a}F_{+-}{}^{b} + \mathrm{e}^{\sigma}(D_+\sigma)(D_-\sigma) - \frac{1}{2}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) \\ &\quad + h\mathrm{e}^{\sigma}\Big\{\frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) - (D_-\sigma)^2\Big\} + 2\mathrm{e}^{\sigma}D_-^2 h. \end{aligned} \tag{3.23}$$

The last term in (3.23) can be expressed as

$$\begin{aligned} \mathrm{e}^{\sigma}D_-^2 h &= \mathrm{e}^{\sigma}\Big(\partial_- - A_-{}^{b}\partial_b\Big)\Big(\partial_- h - A_-{}^{a}\partial_a h\Big) \\ &= \mathrm{e}^{\sigma}\Big\{\partial_-^2 h - \partial_-\Big(A_-{}^{a}\partial_a h\Big) - A_-{}^{a}\partial_a(D_- h)\Big\} \\ &= -(\partial_-\mathrm{e}^{\sigma})(\partial_- h) + (\partial_-\mathrm{e}^{\sigma})\Big(A_-{}^{a}\partial_a h\Big) + \partial_a\Big(\mathrm{e}^{\sigma}A_-{}^{a}\Big)(D_- h) \\ &\quad + \partial_-\Big(\mathrm{e}^{\sigma}\partial_- h\Big) - \partial_-\Big(\mathrm{e}^{\sigma}A_-{}^{a}\partial_a h\Big) - \partial_a\Big(\mathrm{e}^{\sigma}A_-{}^{a}D_- h\Big) \\ &\simeq -\mathrm{e}^{\sigma}(\partial_-\sigma)(D_- h) + \mathrm{e}^{\sigma}A_-{}^{a}(\partial_a\sigma)(D_- h) + \mathrm{e}^{\sigma}(\partial_a A_-{}^{a})(D_- h) \\ &= -\mathrm{e}^{\sigma}(D_-\sigma)(D_- h), \end{aligned} \tag{3.24}$$

where we dropped the surface term and used (3.13c). This can be written as

$$\begin{aligned} \mathrm{e}^{\sigma}(D_-\sigma)(D_- h) &= \mathrm{e}^{\sigma}(D_-\sigma)\Big(\partial_- h - A_-{}^{a}\partial_a h\Big) \\ &= -h\partial_-\Big(\mathrm{e}^{\sigma}D_-\sigma\Big) + h\partial_a\Big(\mathrm{e}^{\sigma}A_-{}^{a}D_-\sigma\Big) + \partial_-\Big(h\mathrm{e}^{\sigma}D_-\sigma\Big) \\ &\quad -\partial_a\Big(h\mathrm{e}^{\sigma}A_-{}^{a}D_-\sigma\Big) \\ &\simeq -h\mathrm{e}^{\sigma}(\partial_-\sigma)(D_-\sigma) - h\mathrm{e}^{\sigma}\partial_-(D_-\sigma) + h\mathrm{e}^{\sigma}A_-{}^{a}(\partial_a\sigma)(D_-\sigma) \\ &\quad + h\mathrm{e}^{\sigma}(\partial_a A_-{}^{a})(D_-\sigma) + h\mathrm{e}^{\sigma}A_-{}^{a}\partial_a(D_-\sigma) \\ &= -h\mathrm{e}^{\sigma}\Big\{D_-^2\sigma + (D_-\sigma)^2\Big\}. \end{aligned} \tag{3.25}$$

We therefore have

$$\mathrm{e}^{\sigma}D_-^2\ h \simeq h\mathrm{e}^{\sigma}\Big\{D_-^2\sigma + (D_-\sigma)^2\Big\}, \tag{3.26}$$

neglecting the surface terms. The resulting (1+1)-dimensional action principle therefore becomes

$$\begin{aligned} \mathcal{L}_2 &= -\frac{1}{2}\mathrm{e}^{2\sigma}\rho_{ab}F_{+-}{}^{a}F_{+-}{}^{b} + \mathrm{e}^{\sigma}(D_+\sigma)(D_-\sigma) - \frac{1}{2}\mathrm{e}^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) \\ &\quad + h\mathrm{e}^{\sigma}\Big\{2D_-^2\sigma + (D_-\sigma)^2 + \frac{1}{2}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd})\Big\}, \end{aligned} \tag{3.27}$$

up to the surface terms. Notice that h is a Lagrange multiplier, whose variation yields the constraint

$$H_0 = D_-^2\sigma + \frac{1}{2}(D_-\sigma)^2 + \frac{1}{4}\rho^{ab}\rho^{cd}(D_-\rho_{ac})(D_-\rho_{bd}) \approx 0. \tag{3.28}$$

From this (1+1)-dimensional point of view, h is the lapse function (or a pure gauge) that prescribes how to 'move forward in the u-time', carrying the surface N_2 at each point of the section $u =$ constant. The constraint, $H_0 \approx 0$, is *polynomial* in σ and $A_-{}^a$, and contains a non-polynomial term of the non-linear sigma model type but in (1+1)-dimensions, where such models often admit exact solutions. This allows us to view the problem of the constraints of general relativity [14] from a new perspective.

We now have the (1+1)-dimensional action principle for the algebraically special class of space-times that contain a twist-free null vector field. It is described by the Yang-Mills action, interacting with the fields σ and ρ_{ab} on the 'flat' (1+1)-dimensional surface, which however must satisfy the constraint $H_0 \approx 0$. (The flatness of the (1+1)-dimensional surface can be seen from the fact that the lapse function, h, can be chosen as zero, provided that $H_0 \approx 0$ holds.) The infinite dimensional group of the diffeomorphisms of N_2 is *built-in* as the local gauge symmetry, via the minimal couplings to the gauge fields.

Having formulated the algebraically special class of space-times as a gauge theory on (1+1)-dimensions, we may wish to apply varieties of field theoretic methods developed in (1+1)-dimensions. For instance, the action (3.27) can be viewed as the bosonized form [15] of *some* version of the (1+1)-dimensional QCD in the infinite dimensional limit of the gauge group [16]. For small fluctuations of σ, the action (3.27) becomes

$$\mathcal{L}_2 = -\frac{1}{2}\rho_{ab}F_{+-}{}^aF_{+-}{}^b + (D_+\sigma)(D_-\sigma) - \frac{1}{2}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}), \tag{3.29}$$

modulo the constraint $H_0 \approx 0$. It is beyond the scope of this article to investigate these theories as (1+1)-dimensional quantum field theories. However, this formulation raises many intriguing questions such as: would there be any phase transition in quantum gravity as viewed as the (1+1)-dimensional quantum field theories? If it does, then what does that mean in quantum geometrical terms? Thus, general relativity, as viewed from the (1+1)-dimensional perspective, renders itself to be studied as a gauge theory in full sense [17], at least for the class of space-times discussed here.

4. w_∞-gravity as special cases

In the previous section we derived the action principle on (1+1)-dimensions as the vantage point of studying general relativity for this algebraically special class of space-times. We now ask different but related questions: what kinds of other (1+1)-dimensional field theories related to this problem can we study? For these, let us consider the case where the local gauge symmetry is replaced by the area-preserving diffeomorphisms of N_2. (For these varieties of field theories, we shall drop the constraint (3.28) for the moment. It is at this point that we are departing from general relativity.) This class of field theories naturally realizes the so-called w_∞-gravity [8, 9] in a linear and geometric way, as we now describe.

The area-preserving diffeomorphisms are generated by the vector fields ξ^a, tangent to the surface N_2 and divergence-free,

$$\partial_a \xi^a = 0. \tag{4.1}$$

Let us find the gauge fields $A_\pm{}^a$ compatible with the divergence-free condition (4.1). Taking the divergence of both sides of (3.10c) and (3.10d), we have

$$\partial_a \delta A_\pm{}^a = -\partial_\pm(\partial_a \xi^a) + \partial_a [A_\pm, \xi]^a. \tag{4.2}$$

This shows that the condition $\partial_a A_\pm{}^a = 0$ is invariant under the area-preserving diffeomorphisms, and characterizes a special subclass of the gauge fields, compatible with the condition (4.1). Moreover, when $\partial_a A_\pm{}^a = 0$, the fields ρ_{ab} and σ behave under the area-preserving diffeomorphisms as a tensor and a scalar field, respectively, as (3.13b) and (3.13c) suggest. Indeed, the Jacobian for the area-preserving diffeomorphisms is just 1, disregarding the distinction between the tensor fields and the tensor densities. The (1+1)-dimensional action principle now becomes

$$\mathcal{L}_2' = -\frac{1}{2} \mathrm{e}^{2\sigma} \rho_{ab} F_{+-}{}^a F_{+-}{}^b + \mathrm{e}^\sigma (D_+\sigma)(D_-\sigma) - \frac{1}{2} \mathrm{e}^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}), \tag{4.3}$$

where $D_\mu \sigma$, $D_\mu \rho_{ab}$, and $F_{+-}{}^a$ are

$$D_\pm \sigma = \partial_\pm \sigma - A_\pm{}^a \partial_a \sigma, \tag{4.4a}$$
$$D_\pm \rho_{ab} = \partial_\pm \rho_{ab} - [A_\pm, \rho]_{ab}, \tag{4.4b}$$
$$F_{+-}{}^a = \partial_+ A_-{}^a - \partial_- A_+{}^a - [A_+, A_-]^a. \tag{4.4c}$$

Under the infinitesimal variations

$$\delta y^a = \xi^a(y, u, v), \qquad \delta x^\mu = 0, \qquad (\partial_a \xi^a = 0), \tag{4.5}$$

the fields transform as

$$\delta\sigma = -[\xi, \sigma] = -\xi^a \partial_a \sigma, \tag{4.6a}$$
$$\delta\rho_{ab} = -[\xi, \rho]_{ab} = -\xi^c \partial_c \rho_{ab} - (\partial_a \xi^c)\rho_{cb} - (\partial_b \xi^c)\rho_{ac}, \tag{4.6b}$$
$$\delta A_+{}^a = -D_+ \xi^a = -\partial_+ \xi^a + [A_+, \xi]^a, \tag{4.6c}$$
$$\delta A_-{}^a = -\partial_- \xi^a, \tag{4.6d}$$

which shows that it *is* a linear realization of the area-preserving diffeomorphisms. The geometric picture of the action principle (4.3) is now clear: it is equipped with the natural bundle structure, where the gauge fields are the connections valued in the Lie algebra associated with the area-preserving diffeomorphisms of N_2. Thus the action principle (4.3) provides a field theoretical realization of w_∞-gravity [8, 9] in a linear and geometric way, with the built-in area-preserving diffeomorphisms as the local gauge symmetry.

With this picture of w_∞-geometry at hands, we may construct as many different realizations of w_∞-gravity as one wishes. The simplest example would be a single real scalar field representation, which we may write

$$\mathcal{L}_2'' = -\frac{1}{2} F_{+-}{}^a F_{+-}{}^a + (D_+\sigma)(D_-\sigma), \tag{4.7}$$

where we used δ_{ab} in the summation, and $D_\pm\sigma$ and $F_{+-}{}^a$ are as given in (4.4a) and (4.4c). By choosing the gauge $A_-{}^a = 0$ and eliminating the auxiliary field A_+^a in terms of σ using the equations of motion of A_+^a, we recognize (4.7) a single real scalar field realization of w_∞-gravity. In presence of the auxiliary field A_+^a, (4.7) provides an example of the *linearized* realization of w_∞-gravity for a single real scalar field. It would be interesting to see if the representation (4.7) is related to the ones constructed in the literatures [8, 9].

5. Discussion

In this review, we examined space-times of 4-dimensions from a (1+1)-dimensional point of view. That general relativity admits such a description is rather surprising, even though the action principle in general appears rather formal. For the algebraically special class of space-times, however, the (1+1)-dimensional action principle, as we have shown here, is formulated as the Yang-Mills type gauge theories interacting with matter fields, where the infinite dimensional group of diffeomorphisms of the 2-surface becomes the 'internal' gauge symmetry. The constraint conjugate to the lapse function appears partly as polynomial. The non-polynomial part is a typical non-linear sigma model type in (1+1)-dimensions, where such models often admit exact solutions. We also discussed the so-called w_∞-gravity as special cases of the algebraically special class of space-times. The detailed study of the w_∞-gravity and its geometry in terms of the fibre bundle will be presented somewhere else.

We wish to conclude with a few remarks. First, one might be interested in finding exact solutions of the Einstein's equations in this formulation. Various two (or more) Killing reductions of the Einstein's equations have been known for sometime which led to the discovery of many exact solutions to the Einstein's equations, by making the system essentially two (or lower) dimensional. In our formulation, the Einstein's equations are already put into a two dimensional form without such assumptions. This might be useful in finding new solutions of the Einstein's equations, which possess no Killing symmetries[5].

[5]Interestingly, there *are* exact solutions of the Einstein's equations which possess no *space-time*

Second, we need to find the constraint algebras for the algebraically special class of space-times explicitly in terms of the variables we used here. As we have shown here, the splitting of the metric variables into the gauge fields and the 'matter fields' is indeed suitable for the description of general relativity as Yang-Mills type gauge theories in (1+1)-dimensions. It remains to study the constraint algebras in detail to see if the ordering problem in the constraints of general relativity becomes manageable in terms of these variables.

Lastly, that the Lie algebra of $SU(N)$ for large N can be used as an approximation of the infinite dimensional Lie algebra of the area-preserving diffeomorphisms of the 2-surface has been suggested as a way of 'regulating' the area-preserving diffeomorphisms. In connection with the problem regarding the regularization of quantum gravity in this formulation, one might wonder as to whether it is also possible to approximate the diffeomorphism algebras of the 2-surface in terms of finite dimensional Lie algebras in a certain limit. There seem to be many interesting questions to be asked about general relativity in this formulation.

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